

A GEOMETRIC STUDY OF $f(r)$ -GRAVITY SPACETIMES THROUGH PROJECTIVE CURVATURE TENSOR

Shyam Kishor and Anjum Ara

Department of Mathematics and Astronomy,
University of Lucknow,
Lucknow - 226007, Uttar Pradesh, INDIA

E-mail : skishormath@gmail.com, anjumara891997@gmail.com

(Received: Apr. 07, 2025 Accepted: Mar. 27, 2026 Published: Apr. 30, 2026)

Abstract: This research investigates spacetime with projective curvature tensor in $f(r)$ gravity. It explores projectively flat and projectively flat perfect fluid spacetime, determining isotropic pressure (p) and energy density (σ). The study extends to energy conditions and reveals that spacetimes with divergence-free projective curvature tensor may represent a dark energy dominated era or have constant isotropic pressure and energy density, especially when the energy-momentum tensor is recurrent or bi-recurrent.

Keywords and Phrases: Spacetime, Perfect Fluid Spacetime, $f(r)$ -Gravity, Projective Curvature Tensor, Energy-momentum tensor.

2020 Mathematics Subject Classification: 53A20, 53C25, 53C50, 83C05, 83C15.

1. Introduction

Projective geometry and projective transformations have deep roots in mathematics, particularly in the field of geometry. Projective geometry originated in the early 19th century, with influential contributions from mathematicians like Jean-Victor Poncelet and August Ferdinand Möbius.

In the context of differential geometry, projective transformations play a significant role in understanding the geometry of curves and surfaces. Differential

geometry extends the study of curves and surfaces in space, and projective transformations can be used to analyze how these shapes change while preserving certain geometric properties. The geometry that deals with a projective transformation is called projective geometry. A spacetime is said to be projectively flat when M is zero, indicating constant curvature. As a result, the deviation of a spacetime from constant curvature is measured by the projective curvature tensor M . The significance of the projective curvature tensor in mathematics and physics has been emphasized by several authors, including Bogdanov, L.V., and Eugene, V. F. [4], Chaubey, S. K. et al. [9], and Haseeb, A. [17].

In general relativity (GR), the Einstein field equations (EFE) are given by

$$R_{ab} - \frac{r}{2}g_{ab} = \kappa T_{ab}, \quad (1.1)$$

where R_{ab} is the Ricci tensor, r is the scalar curvature, g_{ab} is the metric tensor, κ is the gravitational constant, and T_{ab} is the energy-momentum tensor, as described by O'Neill, B. [20]. With the help of these equations, it can be concluded that the energy-momentum tensor T_{ab} is divergence-free. This condition is satisfied whenever $\nabla^a T_{ab} = 0$, where ∇ denotes the covariant derivative with respect to the Levi-Civita connection. A modified theory that extends general relativity is $f(r)$ -gravity. It is a type of modified gravity theory that aims to explain gravity without requiring dark matter or dark energy. Instead, it modifies the way gravity depends on the curvature of spacetime, by replacing the Ricci scalar r with a more general function $f(r)$ in Einstein's field equations. The EFE in $f(r)$ -gravity is given as:

$$\kappa T_{ab} = f'(r)R_{ab} - f''(r)\nabla_a r \nabla_b r - f''(r)\nabla_a \nabla_b r + g_{ab} \left[f'''(r)\nabla_\kappa r \nabla^\kappa r + f''(r)\nabla^2 r - \frac{1}{2}f(r) \right]. \quad (1.2)$$

where $f(r)$ is an arbitrary function of the scalar curvature r and $f'(r) = \frac{df}{dr}$, which must be positive to ensure attractive gravity, as shown by Capozziello, S. et al. [5]. $f(r)$ gravity, as discussed by Sotiriou, T. P., and Faraoni, V. [23], is a well-known modification of general relativity in which the Ricci scalar r is replaced by a general function $f(r)$. As an instance, quintessence and cosmic acceleration in the framework of $f(r)$ gravity were studied by Capozziello, S. [6]. Furthermore, Capozziello, S. et al. established in [7] that within a generalized Robertson-Walker spacetime with divergence-free conformal curvature tensor, the higher-order gravity tensor takes the form of a perfect fluid.

Several recent studies by De, A. et al. [10], De, A. and Loo, T. H. [11, 12] have investigated weakly Ricci symmetric spacetimes $(WRS)_4$, almost pseudo-Ricci symmetric spacetimes $(APRS)_4$, and conformally flat generalized Ricci recurrent spacetimes in the context of $f(r)$ gravity. Motivated by these works, the present

paper aims to study projectively flat spacetimes and projectively flat perfect fluid spacetimes in the context of $f(r)$ gravity.

The paper is organized as follows. Section 2 deals with projectively flat spacetimes in $f(r)$ gravity. Section 3 is devoted to the study of projectively flat perfect fluid spacetimes, along with the analysis of various energy conditions. Section 4 focuses on spacetimes with divergence-free projective curvature tensor in $f(r)$ gravity. Finally, Section 5 concludes the paper with a summary of the main results and potential directions for future research.

2. Projectively Flat Spacetimes in $f(r)$ -Gravity

The projective curvature tensor of type $(0, 4)$ is given by Eastwood, M. [14]:

$$M_{abcd} = R_{abcd} - \frac{1}{(n-1)}[R_{ac}g_{bd} - R_{ad}g_{bc}], \tag{2.1}$$

where R_{abcd} and g_{ab} are the Riemann curvature tensor and the metric tensor, respectively, with a, b, c, d being tensor indices.

It is well known that a spacetime is projectively flat if and only if it is of constant curvature. For completeness, we provide the proof.

Let us consider the case when $M_{abcd} = 0$. Then, from equation (2.1), we have

$$R_{abcd} = \frac{1}{n-1} (R_{ac}g_{bd} - R_{ad}g_{bc}). \tag{2.2}$$

Contracting the above equation with g^{ac} , we obtain

$$R_{bd} = \frac{1}{n-1} (rg_{bd} - R_{bd}), \tag{2.3}$$

which gives

$$R_{bd} = \frac{r}{n}g_{bd}. \tag{2.4}$$

Substituting this into equation (2.2), we obtain

$$R_{abcd} = \frac{r}{n(n-1)} (g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{2.5}$$

Hence, the spacetime is of constant curvature.

Conversely, if a spacetime is of constant curvature, then

$$R_{abcd} = \frac{r}{n(n-1)} (g_{ac}g_{bd} - g_{ad}g_{bc}).$$

which implies that $M_{abcd} = 0$.

Hence, the spacetime is projectively flat.

This leads to the following theorem:

Theorem 2.1. *A spacetime is projectively flat if and only if it is of constant curvature.*

As an immediate consequence, we have the following corollary.

Corollary 2.2. *A projectively flat spacetime is an Einstein manifold and hence has constant scalar curvature.*

Considering Corollary (2.2), the field equation (1.2) in $f(r)$ -gravity becomes

$$R_{ab} - \frac{f}{2f'}g_{ab} = \frac{\kappa}{f'}T_{ab}. \quad (2.6)$$

For the vacuum case, we have

$$R_{ab} - \frac{f}{2f'}g_{ab} = 0. \quad (2.7)$$

Contracting with g^{ab} and integrating the result, we obtain

$$f = \lambda r^{\frac{n}{2}}, \quad (2.8)$$

where λ is a constant.

For the converse case, if we assume equation (2.8) then we get equation (2.7).

Consequently, the following theorem can be stated:

Theorem 2.3. *A necessary and sufficient condition for a projectively flat spacetime in $f(r)$ -gravity to be vacuum is $f = \lambda r^{\frac{n}{2}}$.*

If the following equation

$$L_{\xi}g_{ab} = 0, \quad (2.9)$$

holds, then the vector field ξ is called a Killing vector field. For the equation

$$L_{\xi}g_{ab} = 2\phi g_{ab}, \quad (2.10)$$

the vector field ξ is called a conformal Killing vector field, where L_{ξ} is the Lie derivative with respect to the vector field ξ and ϕ is a scalar function, as discussed by Abu, Donia, H. et al. [1], Mallick, S. and De, U. C. [19], and Zengin, F. O. [25]. The symmetry of a spacetime is measured by the number of independent Killing vector fields the spacetime admits. A spacetime of maximum symmetry has a constant curvature. A spacetime M is said to admit a matter collineation with

respect to a vector field ξ if the Lie derivative of the energy-momentum tensor T with respect to ξ satisfies

$$L_\xi T_{ab} = 0. \tag{2.11}$$

It is clear that every Killing vector field is a matter collineation, but the converse is not generally true. If the Lie derivative of T_{ab} with respect to ξ satisfies the following condition, as discussed by Abu, Donia, H. et al. [1], Mallick, S. and De, U. C. [19], and Zengin, F. O. [25]:

$$L_\xi T_{ab} = 2\phi T_{ab}, \tag{2.12}$$

then the energy-momentum tensor T_{ab} is said to have the Lie inheritance property along the flow lines of the vector field ξ . By inserting equation (2.4) into equation (2.6), we arrive at

$$\left(\frac{r}{n} - \frac{f}{2f'}\right) g_{ab} = \frac{\kappa}{f'} T_{ab}. \tag{2.13}$$

In a spacetime that is projectively flat, the scalar curvature r remains constant, leading to the conclusion that both f and f' are constants. Now, let us consider a non-vacuum projectively flat spacetime M . As a result, the Lie derivative L_ξ of equation (2.13) implies that

$$\left(\frac{r}{n} - \frac{f}{2f'}\right) L_\xi g_{ab} = \frac{\kappa}{f'} L_\xi T_{ab}. \tag{2.14}$$

Suppose the vector field ξ is a Killing vector field in spacetime M , that is, equation (2.9) holds, thus we have

$$L_\xi T_{ab} = 0. \tag{2.15}$$

If equation (2.11) holds true, conversely, we can infer from equation (2.14) that

$$L_\xi g_{ab} = 0. \tag{2.16}$$

Consequently, we are led to state the subsequent theorem:

Theorem 2.4. *Let M be a projectively flat spacetime satisfying $f(r)$ gravity, then the vector field ξ is Killing if and only if M admits matter collineation with respect to ξ .*

The isometry of spacetimes prescribed by Killing vector fields represents a very important type of spacetime symmetry. Spacetimes of constant curvature are known to have maximum such symmetry, that is, they admit the maximum number of linearly independent Killing vector fields. The maximum number of

linearly independent Killing vector fields in an n -dimensional spacetime is $\frac{n(n+1)}{2}$. Additional insights and a detailed discussion on this topic can be found in the works of De, U. C. et al. [13], El-Sayied, H. K. et al. [15], and Shenawy, S., Ünal, B. [22].

From the above theorem and acknowledging this fact, we deduce the following corollary.

Corollary 2.5. *Let M be a non-vacuum projectively flat spacetime satisfying $f(r)$ gravity then M admits the maximum number of matter collineations $\frac{n(n+1)}{2}$.*

Suppose ξ is a conformal Killing vector field, i.e., equation (2.10) holds. Then equation (2.14) implies

$$L_{\xi}T_{ab} = 2\phi T_{ab}. \quad (2.17)$$

If, on the other hand, equation (2.12) holds, then by referring to equation (2.14), we get

$$L_{\xi}g_{ab} = 2\phi g_{ab}. \quad (2.18)$$

This leads us to enunciate the following theorem:

Theorem 2.6. *If M is a projectively flat spacetime satisfying $f(r)$ gravity, then the necessary and sufficient condition for M to be a conformal Killing vector field ξ is that the energy-momentum tensor T_{ab} has the Lie inheritance property along ξ .*

Taking the covariant derivative of both sides of equation (2.13), we get

$$\nabla_{\kappa}T_{ab} = 0. \quad (2.19)$$

As we know, in a projectively flat spacetime r is constant, this implies that f and f' are constant. Substituting equation (2.6) in equation (2.19), we find

$$\nabla_{\kappa}R_{ab} = 0. \quad (2.20)$$

Hence, we can state the following theorem.

Theorem 2.7. *If M is a projectively flat spacetime satisfying $f(r)$ gravity, then M is Ricci symmetric.*

3. Projectively Flat Perfect Fluid Spacetimes in $f(r)$ Gravity

In a four-dimensional space with a perfect fluid, the energy-momentum tensor T_{ab} obeys

$$T_{ab} = (p + \sigma)u_a u_b + pg_{ab}, \quad (3.1)$$

where p is the isotropic pressure, σ is the energy density, and u_a is a unit timelike vector field, as introduced by Blaga, A. M. [3].

Utilizing equation (3.1) in equation (2.6), we obtain

$$R_{ab} = \frac{\kappa}{f'}(p + \sigma) u_a u_b + p g_{ab} + \frac{f}{2f'} g_{ab}. \quad (3.2)$$

Using equation (2.4), we have

$$\frac{r}{4} g_{ab} = \frac{\kappa}{f'} ((p + \sigma) u_a u_b + p g_{ab}) + \frac{f}{2f'} g_{ab}. \quad (3.3)$$

Contracting equation (3.3) with u_a , we get

$$\sigma = \frac{2f - rf'}{4\kappa}. \quad (3.4)$$

Transvecting equation (3.3) with g^{ab} and using equation (3.4), one obtains

$$p = -\frac{(2f - rf')}{4\kappa}. \quad (3.5)$$

Hence, considering the above, we state the following theorem:

Theorem 3.1. *Let M be a projectively flat perfect fluid spacetime admitting $f(r)$ gravity then the isotropic pressure p and the energy density σ are constants and $p = -\frac{(2f - rf')}{4\kappa}$ and $\sigma = \frac{2f - rf'}{4\kappa}$.*

Taking together equation (3.4) and equation (3.5), one easily gets

$$p + \sigma = 0, \quad (3.6)$$

This implies that the spacetime corresponds to a dark energy dominated era, or equivalently, the perfect fluid behaves like a cosmological constant, as described by Stephani, H. et al. [24].

Therefore, we can express the following theorem:

Theorem 3.2. *Let M be a projectively flat perfect fluid spacetime admitting $f(r)$ gravity then, M represents the dark energy dominated era.*

3. Energy Conditions in Projectively Flat Spacetime

In this subsection, we discuss energy conditions in projectively flat spacetimes admitting $f(r)$ gravity. Energy conditions serve as a filtering mechanism for the energy-momentum tensor in both the standard theory of gravity and modified gravity theories. Capozziello, S. et al. [8] examined the weak energy condition (WEC),

dominant energy condition (DEC), null energy condition (NEC), and strong energy condition (SEC) within two extended theories of gravity. To proceed, it is essential to determine the effective isotropic pressure p^{eff} and the effective energy density σ^{eff} , which are required for formulating these energy conditions.

Rewriting equation (2.6) gives:

$$R_{ab} - \frac{1}{2}r g_{ab} = \frac{\kappa}{f'} T_{ab}^{eff}, \quad (3.7)$$

where,

$$T_{ab}^{eff} = T_{ab} + \frac{(f - r f')}{2\kappa} g_{ab}. \quad (3.8)$$

From this, we can rewrite equation (3.1) in the following way:

$$T_{ab}^{eff} = (p^{eff} + \sigma^{eff}) u_a u_b + p^{eff} g_{ab}, \quad (3.9)$$

where, $p^{eff} = p + \frac{(f - r f')}{2\kappa}$ and $\sigma^{eff} = \sigma - \frac{(f - r f')}{2\kappa}$.

Applying both equation (3.4) and equation (3.5) leads to:

$$p^{eff} = -\frac{r f'}{\kappa}, \sigma^{eff} = \frac{r f'}{\kappa}. \quad (3.10)$$

We now examine the energy conditions for an effective matter distribution resembling a perfect fluid in the context of $f(r)$ gravity, as discussed by Capozziello, S. et al. [8] and Santos, J. et al. [21]:

- 1) Null energy condition (NEC): it says that $p^{eff} + \sigma^{eff} \geq 0$.
 - 2) Weak energy condition (WEC): it states that $\sigma^{eff} \geq 0$ and $p^{eff} + \sigma^{eff} \geq 0$.
 - 3) Dominant energy condition (DEC): it states that $\sigma^{eff} \geq 0$ and $|p_{eff}| \leq \sigma_{eff}$.
 - 4) Strong energy condition (SEC): it states that $\sigma^{eff} + 3p^{eff} \geq 0$ and $p^{eff} + \sigma^{eff} \geq 0$.
- In this particular context, the energy conditions mentioned above are consistently valid under the condition $r f' \geq 0$. As discussed earlier, f' must be positive to ensure attractive gravity. Thus, the energy conditions mentioned earlier are satisfied if $r \geq 0$.

3. Divergence-Free Projective Curvature Tensor in $f(r)$ Gravity

The expression for the divergence of the projective curvature tensor for $n = 4$ is given by Ahsan, Z., Siddiqui, S. A. [2]:

$$\nabla_h M_{abc}^h = \nabla_h R_{abc}^h - \frac{1}{3} [g_{bc} \nabla_a R - g_{ac} \nabla_b R]. \quad (4.1)$$

It is well known that

$$\nabla_h R_{abc}^h = \nabla_a R_{bc} - \nabla_b R_{ac}. \quad (4.2)$$

Substituting equation (4.2) into equation (4.1), we get

$$\nabla_h M_{abc}^h = \nabla_a R_{bc} - \nabla_b R_{ac} - \frac{1}{3}[g_{bc} \nabla_a R - g_{ac} \nabla_b R]. \quad (4.3)$$

Assume that the projective curvature tensor is divergence free, that is, $\nabla_h M_{abc}^h = 0$, then

$$\nabla_a R_{bc} - \nabla_b R_{ac} - \frac{1}{3}[g_{bc} \nabla_a R - g_{ac} \nabla_b R] = 0. \quad (4.4)$$

Contracting equation (4.4) with g_{bc} and using $\nabla_b R_a^b = \frac{1}{2} \nabla_a R$, we obtain

$$\nabla_a R = 0. \quad (4.5)$$

Applying equation (4.5) to equation (4.4), we get

$$\nabla_a R_{bc} = \nabla_b R_{ac}. \quad (4.6)$$

Considering equation (4.5), equation (1.2) becomes

$$R_{ab} - \frac{f}{2f'} g_{ab} = \frac{\kappa}{f'} T_{ab}. \quad (4.7)$$

Utilizing equation (4.7) in equation (4.6), we have

$$\nabla_a T_{bc} = \nabla_b T_{ac}. \quad (4.8)$$

This shows that the energy-momentum tensor is of Codazzi type, as discussed by Gray, A. [16]. Thus, we have the following corollary:

Corollary 4.1. *Let M be a spacetime with divergence free projective curvature tensor admitting $f(r)$ gravity then the energy-momentum tensor of M is of Codazzi type.*

If the following condition

$$(\nabla_h \nabla_k - \nabla_k \nabla_h) R_{ab} = 0, \quad (4.9)$$

holds, then the spacetime M is called a Ricci semi-symmetric, as discussed by İnan, Ü. [18].

For the equation,

$$(\nabla_h \nabla_k - \nabla_k \nabla_h) T_{ab} = 0, \quad (4.10)$$

the energy-momentum tensor is called semi-symmetric. From equation (4.7), we have

$$(\nabla_h \nabla_k - \nabla_k \nabla_h) R_{ab} = \frac{\kappa}{f'} (\nabla_h \nabla_k - \nabla_k \nabla_h) T_{ab}. \quad (4.11)$$

Hence, we can assert the following theorem:

Theorem 4.2. *A Necessary and Sufficient condition for a spacetime M with divergence free projective curvature tensor with $f(r)$ gravity to be Ricci semi-symmetric is that the energy-momentum tensor of M is semi-symmetric.*

If there exists a non-zero 1-form λ_k such that

$$\nabla_k T_{ab} = \lambda_k T_{ab} \quad (4.12)$$

then the energy-momentum tensor T_{ab} is called recurrent. On the other hand, the energy-momentum tensor T_{ab} is called bi-recurrent if there exists a non-zero tensor ϵ_{hk} such that

$$\nabla_h \nabla_k T_{ab} = \epsilon_{hk} T_{ab}. \quad (4.13)$$

From the above definition, it follows that every recurrent tensor field is also bi-recurrent. Suppose that the energy-momentum tensor T_{ab} is any $(0, 2)$ symmetric recurrent tensor, that is,

$$\nabla_k T_{ab} = \lambda_k T_{ab}, \quad (4.14)$$

Contracting equation (4.14) with g_{ab} , we get

$$\lambda_k = \frac{1}{T} \nabla_k T, \quad (4.15)$$

where, $T = g^{ab} T_{ab}$.

By covariantly differentiating both sides and in accordance with equation (4.15), we establish

$$\nabla_l \lambda_k + \lambda_k \lambda_l = \frac{1}{T} \nabla_l \nabla_k T. \quad (4.16)$$

Again, by differentiating equation (4.14) covariantly and using equation (4.16), we obtain

$$\nabla_l \nabla_k T_{ab} = \left(\frac{1}{T} \nabla_l \nabla_k T \right) T_{ab}. \quad (4.17)$$

It follows that

$$(\nabla_h \nabla_k - \nabla_k \nabla_h) T_{ab} = 0. \quad (4.18)$$

In the same manner, the same outcome holds for a bi-recurrent $(0, 2)$ symmetric tensor.

In consideration of the above discussion, the following conclusion arises:

Lemma 4.3. *A symmetric tensor of type $(0, 2)$ that is bi-recurrent is semi-symmetric.*

Suppose the energy-momentum tensor T_{ab} is recurrent or bi-recurrent. Therefore, from Lemma-(4.3), we can conclude that T_{ab} is semi-symmetric.

Theorem 4.4. *If the energy-momentum (Ricci) tensor of a spacetime M with divergence free projective curvature tensor admitting $f(r)$ gravity is recurrent or bi-recurrent, then the Ricci (energy-momentum) tensor of M is semi-symmetric.*

If we consider M as a perfect fluid spacetime with divergence free projective curvature tensor whose energy-momentum tensor is recurrent or bi-recurrent then by using equation (3.1) in equation (4.7) we get

$$R_{ab} = \mu g_{ab} + \nu u_a u_b, \tag{4.19}$$

where,

$$\mu = \frac{f}{2f'} + \frac{\kappa p}{f'} = \frac{1}{2f'}(f + 2\kappa p), \tag{4.20}$$

$$\nu = \frac{\kappa}{f'}(p + \sigma). \tag{4.21}$$

Contracting equation (4.19) with g_{ab} , we have

$$r = \frac{1}{f'}[3\kappa p + 2f - \kappa\sigma]. \tag{4.22}$$

Considering the recurrence or bi-recurrence of T_{ab} , it implies $(\nabla_h \nabla_k - \nabla_k \nabla_h)T_{ab} = 0$, subsequently resulting in $(\nabla_h \nabla_k - \nabla_k \nabla_h)R_{ab} = 0$.

Therefore, from equation (4.19), it can be inferred that

$$\nu u_a (\nabla_h \nabla_k - \nabla_k \nabla_h) u_b + \nu u_b (\nabla_h \nabla_k - \nabla_k \nabla_h) u_a = 0. \tag{4.23}$$

Taking contraction with u^a , we get

$$n u (\nabla_h \nabla_k - \nabla_k \nabla_h) u_b + \nu u_b u_a (\nabla_h \nabla_k - \nabla_k \nabla_h) u_a = 0. \tag{4.24}$$

Since $u^a (\nabla_h \nabla_k - \nabla_k \nabla_h) u_a = 0$, thus we have $\nu (\nabla_h \nabla_k - \nabla_k \nabla_h) u_b = 0$.

Equivalently, it is

$$\nu R_{hkb}^m u_m = 0. \tag{4.25}$$

The following cases are implied by this equation:

Case 1- If $R_{hkb}^m u_m \neq 0$, then $\nu = 0$ and hence $p + \sigma = 0$. This indicates that the spacetime corresponds to a dark energy dominated era and the fluid behaves like a cosmological constant.

Case 2- If $\nu \neq 0$, then $R_{hkb}^m u_m = 0$, hence a contraction with g^{hb} implies that $R_{ma} u^m = 0$.

Taking contraction of equation (4.19) with u^a and using $R_{ma} u^m = 0$, we have

$$(\mu - \nu)u_a = 0. \quad (4.26)$$

Then $\mu - \nu = 0$. Making use of equation (4.20), we obtain

$$\sigma = \frac{f}{2\kappa}. \quad (4.27)$$

Utilizing equation (4.27) in equation (4.22), we have

$$p = \frac{(2rf' - 3f)}{6\kappa}. \quad (4.28)$$

Hence, we state the following theorem:

Theorem 4.5. *Let M be a spacetime with divergence free projective curvature tensor admitting $f(r)$ gravity. If the energy-momentum tensor of M is recurrent or bi-recurrent. Then,*

1. *The spacetime represents the dark energy dominated era, or*
2. *The isotropic pressure p and the energy density σ are constants.*

5. Conclusion

The present study demonstrates that the projective curvature tensor plays a significant role in determining the geometric and physical structure of $f(r)$ -gravity spacetimes. In particular, projective flatness enforces constant curvature and significantly constrains the behavior of the energy-momentum tensor.

Furthermore, the analysis indicates that under these geometric conditions, the matter content effectively behaves like a cosmological constant, reflecting a dark energy dominated era. The divergence-free condition further reveals a structured behavior of the energy-momentum tensor, emphasizing its geometric compatibility.

Thus, the study shows that the projective curvature tensor provides an effective tool for linking geometric properties of spacetime with physically meaningful models in modified theories of gravity.

References

- [1] Abu, Donia, H., Shenawy, S., Syied, A. A., The W^* - Curvature Tensor on Relativistic Space-Times, Kyungpook Math. J., 60 (2020), 185–195.

- [2] Ahsan, Z., Siddiqui, S. A., Conircular Curvature Tensor and Fluid Spacetimes, *Int. J. Theor. Phys.*, 48 (2009), 3202–3212.
- [3] Blaga, A. M., Solitons and Geometrical Structures in a Perfect Fluid Spacetime, *Rocky Mountain J. Math.*, 50 (2020), 41–53.
- [4] Bogdanov, L. V., and Eugene, V. F., Projective differential geometry of higher reductions of the two-dimensional Dirac equation, *Journal of Geometry and Physics*, 52 (2002), 328-352.
- [5] Capozziello, S., Mantica, C. A. and Molinari, L. G., General Properties of $F(r)$ Gravity Vacuum Solutions, *Int. J. Mod. Phys. D*, 19(13) (2020).
- [6] Capozziello, S., Curvature Quintessence, *Int. J. Mod. Phys. D*, 11 (2002), 483-492.
- [7] Capozziello, S., Mantica, C. A., Molinari, L. G., Cosmological Perfect Fluids in Higher-Order Gravity, *Gen. Relativity Gravitation*, 52(4) (2020), 36.
- [8] Capozziello, S., Lobo, F. S. N., Mimoso, J. P., Generalized Energy Conditions in Extended Theories of Gravity, *Phys. Rev. D*. (2015), <https://doi.org/10.48550/arXiv.1407.7293>.
- [9] Chaubey, S. K., Kanaujia, R. B, Pankaj, and Yadav, S. K., Projective Curvature Tensor of Riemannian Manifolds Admitting A Projective Semi-Symmetric Connection, *Universal J. of Mathematics and Appl.*, 3(2) (2020), 78-85.
- [10] De, A., Loo, T. H., Arora, S., Sahoo, P., Energy Conditions for a $(WRS)_4$ Spacetime in $F(r)$ -Gravity, *The Eur. Phys. J. Plus*, 136(2) (2021), 218.
- [11] De, A., Loo, T. H., Almost Pseudo Ricci Symmetric Spacetime Solutions in $F(r)$ -Gravity, *Gen. Relativ. Gravit*, 53(5) (2021).
- [12] De, A., Loo, T. H., Solanki, R., Sahoo, P. K., A Conformally Flat Generalized Ricci Recurrent Spacetime in $F(r)$ -Gravity, *Phys Scr.*, 96(8) (2021).
- [13] De, U. C., Shenawy, S., Ünal, B., Sequential Warped Products: Curvature and Conformal Vector fields, *Filomat*, 33 (2019), 4071–4083.
- [14] Eastwood, M., Notes on Projective Differential Geometry, *The IMA Volumes in Mathematics and its Applications*, 144 (2008), 41-60.

- [15] El-Sayied, H. K., Shenawy, S., Syied, N., On Symmetries of Generalized Robertson-walker Spacetimes and Applications, *J. Dynamical Syst. Geometric Theories*, 15 (2017), 51–69.
- [16] Gray, A., Einstein-like Manifolds which Are Not Einstein, *Geometriae dedicata*, 7 (1978), 259–280.
- [17] Haseeb, A., Some results on projective curvature tensor in an ϵ -Kenmotsu manifold, *Palestine J. of Mathematics*, 6(II) (2017), 196–203.
- [18] İnan, Ü., On Metric Contact Pairs with Certain Semi-symmetry Conditions, *Politeknik Dergisi*, 24(1) (2021), 333–338.
- [19] Mallick, S., De, U. C., Spacetimes Admitting W_2 -Curvature Tensor, *Int. J. Geom. Methods Mod. Phys.*, 11(4) (2014).
- [20] O’Neill, B., *Semi-semi-Riemannian Geometry with Applications to Relativity*, New York-London: Academic Press, 103 (1983).
- [21] Santos, J., Alcaniz, J., Reboucas, M., Carvalho, F., Energy Conditions in F(r)-Gravity, *Phys. Rev. D.*, 74 (2007).
- [22] Shenawy, S., Ünal, B., 2-killing Vector fields on Warped Product Manifolds, *Int. J. Math.*, 26(9) (2015).
- [23] Sotiriou, T. P., and Faraoni, V., f(R) Theories Of Gravity, *Rev. Mod. Phys.*, 82 (2010), 451-497.
- [24] Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C., Herlt, E., *Exact Solutions of Einstein’s Field Equations*, Cambridge, United Kingdom: Cambridge University Press (2009), <https://doi.org/10.1017/CBO9780511535185>.
- [25] Zengin, F. O., M-projectively Flat Spacetimes, *Math Rep.*, 14 (2012), 363–370.